# Numerical Methods in The Theory of Topological Solitons

L. Stoica

Faculty of Physics
Universities of Bucharest

International Student Practice, 2018

Bogoliubov Laboratory of Theoretical Physics

Project Supervisor - Prof. Ya. Shnir

# Outline

1. Introduction
   - What are Solitons?
   - Derrick's Theorem
   - The Models Studied
     - O(3) Sigma Model
     - Skyrme Model
2. My Activity
   - O(3) Sigma Model
   - Skyrme Model

# Outline

# What are Solitons?

## Definition

Solitons are stable, localized configurations of the fields that emerge in numerous nonliniar systems.

Such as:

- Nonliniar Optics
- Condensed Mater
- Nuclear Physics
- Cosmology
- Supersymmetric Theories

# Outline

1. Introduction
   - What are Solitons?
   - Derrick's Theorem
   - The Models Studied
     - O(3) Sigma Model
     - Skyrme Model
2. My Activity
   - O(3) Sigma Model
   - Skyrme Model

## Theorem

There are no static, finite energy solutions of the model with the Lagrangian:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)$$

If

$$\frac{dE(\lambda)}{d\lambda}\ |_{\lambda=1} < 0$$

&

$$dim > 2$$

# Implication

The essence of the theorem:
If the solutions are scale invariant (invariant with respect to deformations) they can shrink or expand the configuration indefinetly(i.e. not stable).

This theorem holds the answer to which models will have soliton solutions and which will not.
The models that we will construct will have this in "mind".

# Implication

The essence of the theorem:
If the solutions are scale invariant (invariant with respect to deformations) they can shrink or expand the configuration indefinetly(i.e. not stable).

This theorem holds the answer to which models will have soliton solutions and which will not.
The models that we will construct will have this in "mind".

# Outline

# The Models Studied

- O(3) Sigma Model
- Skyrme Model

# The Models Studied

## O(3) Sigma Model

The most general Lagrangian of the Sigma Model in d+1 dimensions (we will restrict our attention to 2+1 dimensions) is :

$$L = \frac{1}{2}\int d^d x g_{ab}\partial_\mu\phi^a\partial^\mu\phi^b$$

Taking a look at the equations of motion

$$\partial_\mu\partial^\mu\phi^a + \Gamma^a_{bc}\partial_\mu\phi^b\partial^\mu\phi^c = 0$$

# The Models Studied
## O(3) Sigma Model

The most general Lagrangian of the Sigma Model in d+1 dimensions (we will restrict our attention to 2+1 dimensions) is :

$$L = \frac{1}{2}\int d^d x g_{ab}\partial_\mu \phi^a \partial^\mu \phi^b$$

**Taking a look at the equations of motion**

$$\partial_\mu \partial^\mu \phi^a + \Gamma^a_{bc}\partial_\mu \phi^b \partial^\mu \phi^c = 0$$

# The Models Studied
## O(3) Sigma Model

### By the symmetries of the model we find that

$$M = \frac{SO(N)}{SO(N-1)} = S^{N-1}$$

$$M = \frac{SU(N+1)}{SU(N)xU(1)} = CP^n$$

### Restricting our attention to 2+1 dimension

Because of the isomorphism between $CP^1 \& S^2$.
So, $S^2$ admits a complex structure =>$S^2 = CP^1$

# The Models Studied
## O(3) Sigma Model

Reformulating the model in terms of the complex variable $W = \frac{\phi_1 + i\phi_2}{1-\phi_3}$ and making use of the stereographic projection from the North Pole of $S^2$ we can find the components of the field $(\phi_1, \phi_2, \phi_3)$

**We can find the energy and the topological charge of the confinuration in terms of the complex variable**

$$E = \int \frac{\mid W_z \mid^2 + \mid W_{\bar{z}} \mid^2}{(1+ \mid W \mid^2)^2} dz d\bar{z}$$

$$Q = \frac{1}{4\pi} \int \frac{\mid W_z \mid^2 - \mid W_{\bar{z}} \mid^2}{(1+ \mid W \mid^2)^2} dz d\bar{z}$$

# The Models Studied

## O(3) Sigma Model

**The absolute minimum of the energy funtional coresponds to:**

$$W_{\bar{z}} = 0 \rightarrow Q = 4\pi E$$

$$W_z = 0 \rightarrow Q = -4\pi E$$

In both cases we find that the energy functional hits it's minimum if $W(z)$ is a holomorphic funtion or an anti-holomorphic function, depending on the charge sign.

# The Models Studied

## O(3) Sigma Model.The Rational Map Approach.

### The Rational Map

Every holomorphic/anti-holomorphic funtion will satisfy the field equations. To construct a soliton configuartion we consider the most general form of a holomorphic fuction.

$$W(z) = \frac{P(z)}{Q(z)}$$

Where $P(z)$ and $Q(z)$ are polynomials of, at most, degree N.

# The Models Studied
## O(3) Sigma Model.The Rational Map Approach.

### Applying The Rational Map

Making use of the rational map we can construct soliton consigurations, centered at some point $z_0$ with the size and shape controled by the parameters of the polinomial .
(This approach will come in handy in the next model as well.)

# The Models Studied
## Skyrme Model

### Historically

One of the first field models to support soliton solutions.

### Main Idea

Skyrme's idea was to consider baryons as solitons with the identification of the baryon number and the topological charge of the field configuration.
A truly revolutionary idea, as this idea is at the core of some braches of modern theoretical physics up to date.

# The Models Studied
## Skyrme Model

### Historically

One of the first field models to support soliton solutions.

### Main Idea

Skyrme's idea was to consider baryons as solitons with the identification of the baryon number and the topological charge of the field configuration.
A truly revolutionary idea, as this idea is at the core of some braches of modern theoretical physics up to date.

# The Models Studied
## Skyrme Model

The field of the Skyrme model in 3+1 dimensions is the unitary, unimodular matrix $U(\mathbf{r}, t) \in SU(2)$ , $U^\dagger U = I$ .
Which can be written as an expanstion in the quarted of scalar fields $(\pi_0, \pi_1, \pi_2, \pi_3)$, restricted to the surface of the sphere $S^3$.

$$U = \pi^\nu \tau^\nu \longrightarrow_{r \to \infty} I$$

$\nu = 0, 1, 2, 3$
$\tau^\nu$ - the usual Pauli matrices together with the identity matrix
$\vec{\tau} = (I, \sigma^1, \sigma^2, \sigma^3)$

# The Models Studied
## Skyrme Model

### Constructing The Skyrme Model Lagrangian

As usual it includes a quadratic term.

$$L_2 = -\frac{f_\pi^2}{16} tr[(U^\dagger \partial_\mu U)(U^\dagger \partial^\mu U)]$$

Taking into consideration Derrick's theorem we have to add a term of forth order is derivatives.

$$L_4 = \frac{1}{32e^2} tr[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger)]^2$$

This particular choice yield terms in the equations of motion which still are of second order, thus the model remains Lorents invariet. The total lagrangian is $L = L_2 + L_4$

# The Models Studied
## Skyrme Model

Note that, in order to give the triplet of pions mass, one has to supplement the Lagrangian with a symmetry-breaking potential term.

$$L_0 = \frac{m_\pi^2 f_\pi^2}{8} tr[U - I]$$

This is a physical requirment and it has nothing to do with Derrick's theorem.

The final Lagrangian beeing:

$$L = L_4 + L_2 + L_0$$

# The Models Studied
## Skyrme Model

### The equations of motion are:

$$\partial_\mu\{(\partial^\mu U)U^\dagger + \frac{1}{4}[(\partial^\nu U)U^\dagger, [(\partial_\nu U)U^\dagger, (\partial^\mu U)U^\dagger]]\} = \frac{m^2}{2}(U - U^\dagger)$$

### Solving the equations

There are no know analitical solutions to this equations. The only way to obtain these field configuations in all topological sectors is to implement various numerical methods to minimize the coresponding energy funtional.

# The Models Studied
## Skyrme Model

### The equations of motion are:

$$\partial_\mu\{(\partial^\mu U)U^\dagger + \frac{1}{4}[(\partial^\nu U)U^\dagger,[(\partial_\nu U)U^\dagger,(\partial^\mu U)U^\dagger]]\} = \frac{m^2}{2}(U - U^\dagger)$$

### Solving the equations

There are no know analitical solutions to this equations. The only way to obtain these field configuations in all topological sectors is to implement various numerical methods to minimize the coresponding energy funtional.

# The Models Studied

## Skyrme Model.Construction Of Multi-Skyrmeons

### Approximations

- Product Ansatz
- Holonomy of the $SU(2)$ Yang-Mills instantons
- Rational Map Approximation

Later works have shown that the multi-soliton solutions of the Skyrme model posses discrete symmetries. More precisly, skyrmeons of charge $Q = 1 - 4$ have the symmetries of platonic solids and the higher order topological degrees have symmetries of the dihedral group $D_n$ , the extended dihedral groups $D_{nh}$ or $D_{nd}$ or even the icosahedral group $I_h$.

# The Models Studied

## Skyrme Model.Construction Of Multi-Skyrmeons

Because of those symmetries we will focus our attention on the Rational Map Aproximation. The reason beeing, with this approximation we can chose a polynomial subject to the symmetries that we want. This conveys an easy method to get the first few low topological order skyrmeons.

# The Models Studied

## Skyrme Model.Construction Of Multi-Skyrmeons

### The Rational Map Is Defined Almost Identically

The complex variable:

$$z = \frac{x_1 + ix_2}{1 + x_3} = tan\left(\frac{\theta}{2}\right) e^{i\phi}$$

The Rational Map:

$$W(z) = \frac{P(z)}{Q(z)}$$

$P(z)$ and $Q(z)$ polynomials of at most degree N, in z.

# The Models Studied

## Skyrme Model.Construction Of Multi-Skyrmeons

**The energy and the topological charge are given by:**

$$E = \int f'(r)^2 + 2(1 + f'(r)^2)\frac{sin^2(f)}{r^2}\left(\frac{1+\mid z\mid^2}{1+\mid W\mid^2}\left|\frac{dW}{dz}\right|\right)^2$$

$$+\frac{sin^4(f)}{r^4}\left(\frac{1+\mid z\mid^2}{1+\mid W\mid^2}\left|\frac{dW}{dz}\right|\right)^4\frac{2idzd\bar{z}}{(1+|z|^2)^2}$$

$$Q = \frac{1}{4\pi}\int\left(\frac{1+\mid z\mid^2}{1+\mid W\mid^2}\left|\frac{dW}{dz}\right|\right)^2\frac{2idzd\bar{z}}{(1+|z|^2)^2}$$

# Outline

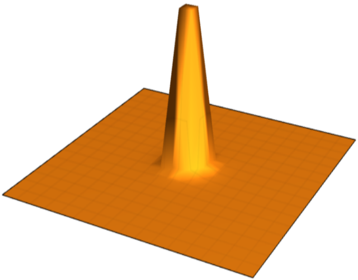
# O(3) Sigma Model.The North Pole Projection

The triplet of scalar fields $\phi^a$ where found in terms of complex variable $W(z)$.

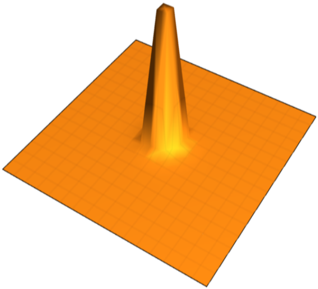
$$(\phi^1, \phi^2, \phi^3) = (\frac{W + \bar{W}}{1 + W\bar{W}}, i\frac{\bar{W} - W}{1 + W\bar{W}}, \frac{1 - W\bar{W}}{1 + W\bar{W}})$$

And plotted the energy density distribution and topological charge density distribution for various configurartions using the rational map $W(z)$.

# O(3) Sigma Model.The North Pole Projection

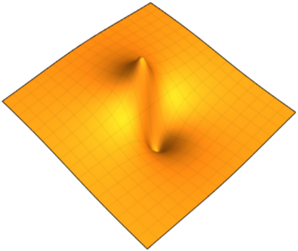


This is the energy density distribution of a soliton centered at (0,0)



This is the charge density distribution of a soliton centered at (0,0)

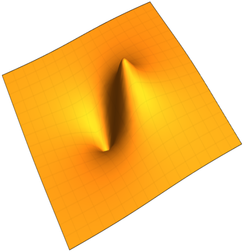
# O(3) Sigma Model.The North Pole Projection



This is the first component of field:

$$\phi^1(x,y)$$

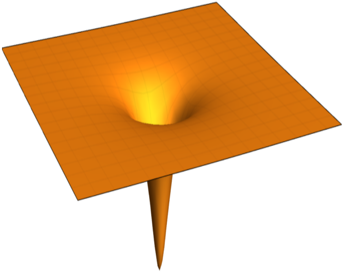
# O(3) Sigma Model.The North Pole Projection



This is the second component of field:

$$\phi^2(x, y)$$

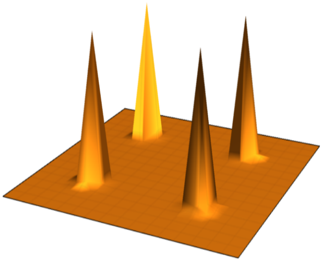
# O(3) Sigma Model.The North Pole Projection

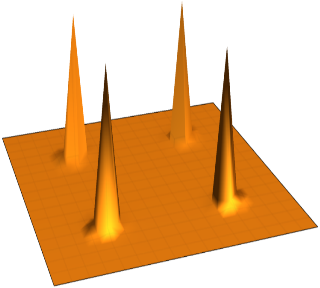


This is the third component of field:

$$\phi^3(x, y)$$

# O(3) Sigma Model.The North Pole Projection





This is the energy density distribution of a 4 soliton configuration centered at (-50,-50),(-50,50),(50,50),(50,-50).

This is the charge density distribution of a 4 soliton configuration centered at (-50,-50),(-50,50),(50,50),(50,-50).

# O(3) Sigma Model.The North Pole Projection

The maps are given by

$$W(z) = \frac{1}{z}$$

for the 1 soliton configuration and by

$$W(z) = \frac{1}{z - (-50 - i50)} + \frac{1}{z - (50 - i50)} + \frac{1}{z - (-50 + i50)} + \frac{1}{z - (50 + i50)}$$

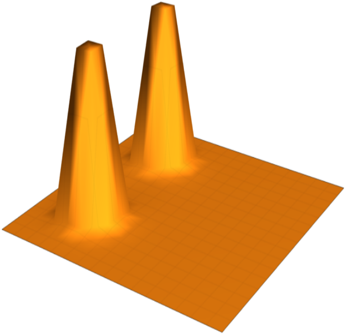
for the 4 soliton configuration.

# O(3) Sigma Model.The South Pole Projection

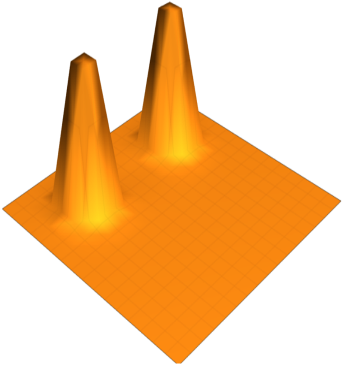
I found the triplet of scalar fields $\phi^a$ in terms of the complex variable using the stereographis projection map from the south pole

$$(\phi^1, \phi^2, \phi^3) = (\frac{W + \bar{W}}{1 + W\bar{W}}, i\frac{\bar{W} - W}{1 + W\bar{W}}, \frac{W\bar{W} - 1}{1 + W\bar{W}})$$

# O(3) Sigma Model.The South Pole Projection

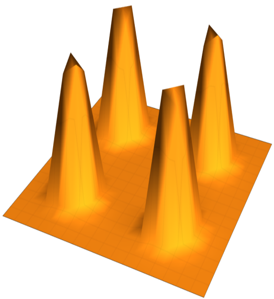


This is the energy density distribution of two solitons centered at (-50,-50),(-50,50)

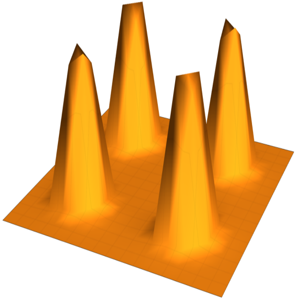


This is the charge density distribution of two solitons centered at (-50,-50),(-50,50)

# O(3) Sigma Model.The South Pole Projection



This is the energy density distribution of a 4 soliton configuration centered at (-50,-50),(-50,50),(50,50),(50,-50).



This is the charge density distribution of a 4 soliton configuration centered at (-50,-50),(-50,50),(50,50),(50,-50).

# O(3) Sigma Model.The South Pole Projection

The rational maps of the configurations are:

$$W(z) = [z - (-50 - i50)] * [z - (-50 + i50)]$$

for the two soliton configuration.

$$W(z) = \prod_{i=1}^{4} (z - z_{0i})$$

For the four soliton configuation.
$z_{0i}$ are the centers of each soliton.

# O(3) Sigma Model

**Remark**

As we can see, the energy density plot and the topological charge density plot match. This is what we would expect to happen and it happens for most of the non-linear theories. Although in general this is not the case.

# Outline

# Skyrme Model

Keeping the last remark in mind, i will continue to show only the topological charge density, the energy density beeing in essence the same thing for the Skyrme Model.

# Skyrme Model

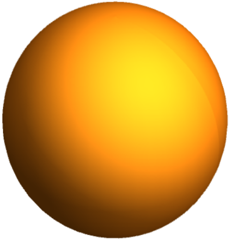
## Reminding The Energy And Topological Charge:

$$E = \int f'(r)^2 + 2(1 + f'(r)^2)\frac{sin^2(f)}{r^2}\left(\frac{1+\mid z\mid^2}{1+\mid W\mid^2}\left|\frac{dW}{dz}\right|\right)^2$$

$$+\frac{sin^4(f)}{r^4}\left(\frac{1+\mid z\mid^2}{1+\mid W\mid^2}\left|\frac{dW}{dz}\right|\right)^4\frac{2idzd\bar{z}}{(1+|z|^2)^2}$$

$$Q = \frac{1}{4\pi}\int\left(\frac{1+\mid z\mid^2}{1+\mid W\mid^2}\left|\frac{dW}{dz}\right|\right)^2\frac{2idzd\bar{z}}{(1+|z|^2)^2}$$

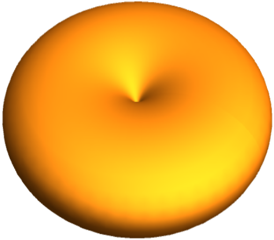
# Skyrme Model.Skyrmeon of charge 1



The rational map for this configuration is:

$$W(z) = z$$

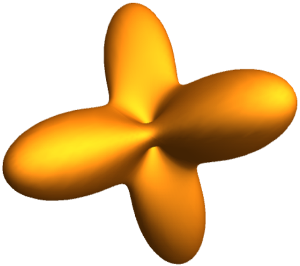
# Skyrme Model.Skyrmeon of charge 2



The rational map for this configuration is:

$$W(z) = z^2$$

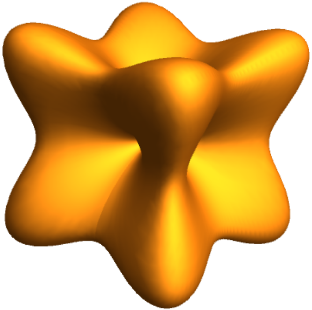
# Skyrme Model.Skyrmeon of charge 3



The rational map for this configuration is:

$$W(z) = \frac{i\sqrt{3}z^2 - 1}{z(z^2 - i\sqrt{3})}$$

# Skyrme Model.Skyrmeon of charge 4



The rational map for this configuration is:

$$W(z) = \frac{z^4 + 2i\sqrt{3}z^2 + 1}{z^4 - 2i\sqrt{3}z^2 + 1}$$

# Skyrme Model.Skyrmeon of charge 7

The rational map for this configuration is:

$$W(z) = \frac{z^5 + \frac{1}{7}}{z^2(-\frac{z^5}{7} + 1)}$$

# Thank You For Your Attention

Thanks to:
Yakov Shnir - For Guidance And Support
Alexandru Jipa - For The Opportunity