Scalar field dynamics in black hole backgrounds

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Statement of the problem

- Within General Relativity one can study characteristic oscillations of black holes (BH) called quasinormal modes (QNMs).
- BH QNMs frequencies are excited by external fields but depend only on BH parameters (mass, charge, angular momentum).
- QNMs appear in the analysis of linear (small) perturbations $\delta g_{\mu\nu}$ of a fixed BH background space-time metric $g^0_{\mu\nu}$:

$$g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}.$$

- The metric perturbations $\delta g_{\mu\nu}$ obey a wave equation (2-nd order partial differential eq.).
- In most cases one can separate variables in the wave eq. and reduce it to a set of linear ordinary differential equations of the form of Schrödinger eq.:

$$\Psi''(x) + \left[\omega^2 - V(x)\right] \Psi(x) = 0.$$

- The *complex* eigenvalues ω^2 are BH QNMs provided that solutions obey certain boundary conditions.

Objectives

- Introducing necessary notions and tools for the above-stated problem.
- Studying an example of that problem, namely, a scattering of a scalar field on a Schwarzschild background.
- Deriving an equation determining the Schwarzschild BH QNMs.

Classical fields in Minkowski space-time

- Minkowski space-time

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$$
$$x \cdot y \equiv \eta_{\mu\nu} x^\mu y^\nu$$

$\mu, \nu = (0, 1, 2, 3)$

- Lorentz group

— Lorentz matrix

$$\Lambda \equiv [\Lambda^\mu{}_\nu]$$

— Defining property

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$$

$\Leftarrow x' \cdot y' = x \cdot y$

— Inverse Lorentz matrix

$$\left(\Lambda^{-1}\right)^\mu{}_\nu = \Lambda_\nu{}^\mu$$

- A field in a Minkowski space-time is a quantity that is a function of the space-time point x^μ.
- Different types of fields can be defined due to their transformation properties w.r.t. the Lorentz transformations, e.g.:

— scalar field

$$\Phi'(x') = \Phi(x)$$

— vector field

$$\mathrm{A}'^\mu(x') = \Lambda^\mu{}_\alpha \mathrm{A}^\alpha(x)$$

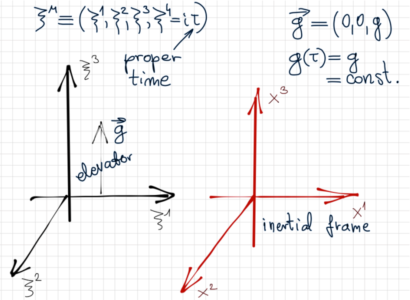
— tensor field

$$\mathrm{T}'^{\mu\nu}(x') = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \mathrm{T}^{\alpha\beta}(x)$$

$x' \equiv x'^\mu = \Lambda^\mu{}_\alpha x^\alpha$.

Introduction to General Relativity

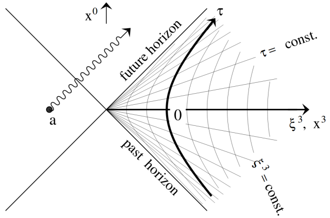
Constantly accelerated elevator



For $\tau = 0$, $\vec{\xi} = \vec{x}$; functions $x^\mu(\xi)$ describe the motion of the elevator

$$x^\mu(\vec{\xi}, \mathrm{i}\tau) = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \cosh(\frac{g\tau}{c})\left(\xi^3 + \frac{c^2}{g}\right) - \frac{c^2}{g} \\ \mathrm{i}\sinh(\frac{g\tau}{c})\left(\xi^3 + \frac{c^2}{g}\right) \end{pmatrix}$$

Rindler space $(x^0 \equiv x^4)$



$$\begin{aligned} g_{\mu\nu} &= \mathrm{diag}(1,1,1,1), \\ ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = \tilde{g}_{\mu\nu}(\xi)d\xi^\mu d\xi^\nu \\ &= (d\vec{\xi})^2 + \left(1 + \frac{g\xi^3}{c^2}\right)^2 (d\xi^4)^2 \end{aligned}$$

- Equivalence Principle: a uniform gravitational field is equivalent to a uniform acceleration.
- Metrics in curved space-times describe gravitational field.

Fields in General Relativity

Scalar, vector, tensor fields are defined w.r.t. general coordinate transformations (one-to-one mappings onto another set of coordinates):

$$x^\mu \Leftrightarrow u^\nu \text{ such that exist } x^\mu{}_{,\nu} = \frac{\partial x^\mu(u)}{\partial u^\nu} \text{ and } u^\nu{}_{,\mu} = \frac{\partial u^\nu(x)}{\partial x^\mu} \;,$$

(in a very special case $x^\mu{}_{,\nu} =$ Lorentz matrix $\in$ Lorentz group).

— Metric tensor $g_{\mu\nu}(x)$:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

— Affine connection (not a tensor!):

$$\Gamma^\lambda_{\mu\nu} = g^{\lambda\alpha}\Gamma_{\alpha\mu\nu}$$

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right)$$

— Riemmann curvature tensor:

$$R^\nu_{\kappa\lambda\alpha} = \partial_\lambda \Gamma^\nu_{\kappa\alpha} - \partial_\alpha \Gamma^\nu_{\kappa\lambda} + \Gamma^\nu_{\lambda\sigma}\Gamma^\sigma_{\kappa\alpha} - \Gamma^\nu_{\alpha\sigma}\Gamma^\sigma_{\kappa\lambda}$$

— Ricci tensor:

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$$

— Ricci scalar:

$$R = g^{\mu\nu}R_{\mu\nu} = R^\mu_\mu$$

— Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu}$$

Einstein´s field equations

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -\frac{8\pi \mathrm{G}_N}{c^4}\, T_{\mu\nu}}$$

- — gravitational constant G_N
- — speed of light c
- — gravitational energy–momentum tensor $T_{\mu\nu}$ — a source of the gravitational field (gravitating matter)

- If $T_{\mu\nu} = 0 \;\Rightarrow\;$ vacuum field equations:

$$\boxed{R_{\mu\nu} = 0}$$

- Vacuum solutions (examples):
  - — flat Minkowski space-time
  - — Schwarzschild solution
  - — Kerr solution

Schwarzschild solution

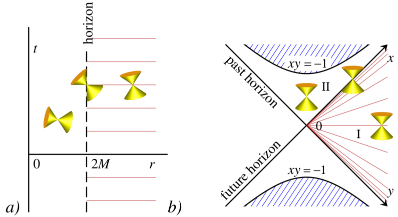
a) in spherical coordinates $(x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)$:

$$
\begin{aligned}
ds^2 &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \\
f(r) &= 1 - \frac{2M}{r}
\end{aligned}
$$

b) in Kruskal-Szekeres coordinates $(t, r, \theta, \varphi) \to (x, y, \theta, \varphi)$ defined by

$$
xy = \left(\frac{r}{2M} - 1\right)\exp\left(\frac{r}{2M}\right), \qquad \frac{x}{y} = \exp\left(\frac{t}{2M}\right),
$$

$$
ds^2 = \frac{32M^3}{r}\exp\left(-\frac{r}{2M}\right)dxdy + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2
$$



The Schwarzschild metric of the empty space surrounding a static massive object with the mass m and $M = G_N m \; (c = 1)$.

Scalar field in Schwarzschild background

- Klein-Gordon equation in Schwarzschild metric:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi\right) = 0 \Rightarrow$$

$$-f(r)^{-1}\partial_t^2\Phi + \frac{1}{r^2}\left[\partial_r\left(f(r)r^2\partial_r\Phi\right)\right] + \frac{1}{r^2\sin\theta}\left[\partial_\theta\left(\sin\theta\partial_\theta\Phi\right)\right] + \frac{1}{r^2\sin\theta}\partial_\varphi^2\Phi$$
$$= 0.$$

- Substitution:

$$\Phi(t,r,\theta,\varphi) = \sum_{\ell,m}\exp(-\mathrm{i}\omega t)Y_{\ell,m}(\theta,\varphi)\frac{\psi(r)}{r}$$

⇒ radial equation for $\psi(r)$:

$$\left(\partial_{r*}^2 - V(r) + \omega^2\right)\psi(r) = 0\,,$$
$$V(r) = \left(1-\frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2}+\frac{2M}{r^3}\right],$$
$$r^* = r + 2M\ln(r-2M)\,.$$

Conclusions

1. We have derived the Schrödinger-like radial equation determining the Schwarzschild BH QNMs excited by the scalar field.

   Calculation of its exact analytic solutions determining ω^2 is an open task which motivates searching for new methods of solutions of such class of equations.

2. The studied problem of the scattering of the scalar field on the Schwarzschild background has fascinating extensions to more astrophysicaly relevant examples, where a comparison with observations is possible!

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Thank you for your attention!