Numerical Methods in Theory of Topological Solitons

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Overview

- Mathematically, topological solitons are finite energy particle-like solutions of nonlinear differential equations, in which the number of particles is a topological quantity.

Introduction

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- Physically, topological solitons are a way of storing a localized lump of energy in a nonlinear system.

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- Topological solitons have applications in a range of areas in particle physics, condensed matter physics, nuclear physics and cosmology.
- One of the simplest examples of solitons is the class of the kink configurations, which appears in the $(1+1)$ dimensional models with a potential possessing two or more degenerated minima.

The ϕ^4 model arises in many different physical situations, it serves as a prototype for many non-linear systems. Indeed, it has a number of applications in condensed matter physics. Furthermore, the ϕ^4 model has been applied in biophysics to describe soliton excitation in DNA double helices.

Lagrangian of ϕ^4 model

$$L = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - \frac{1}{2}(\phi^2 - 1)^2$$

Where the potential is $V(\phi) = \frac{1}{2}(\phi^2 - 1)^2$ and $\phi = \phi(t, x)$ is the field function.

Using Euler Lagrange equations in the field theory for the field ϕ

Euler Lagrange equations

$$\frac{\delta L}{\partial \phi} - \frac{\partial}{\partial_{q^\mu}}\left(\frac{\delta L}{\partial \phi_{,\mu}}\right) = 0$$

Where $\mu = 0, 1, q^0 = t, q^1 = x$ and $\phi_{,\mu} = \partial_{q^\mu}\phi$ So we have

Corresponding field equation

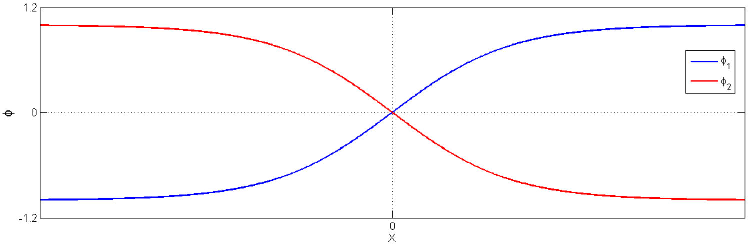
$$\partial_t^2\phi - \partial_x^2\phi + 2\phi(\phi^2 - 1) = 0$$

The kink configuration topological nontrivial static solution of the last equation i.e. $(\partial_t\phi = 0)$.

We can write $\partial_x^2\phi = 2\phi(\phi^2 - 1)$ where the kink interpolate between two vaccua $\phi(\mp\infty) = \pm 1$ and the field is centered around x_0, so we have

Kink and Anti-Kink

$$\phi_{1,2} = \pm tanh(x - x_0)$$



The ϕ^6 model in $(1+1)$-dimensional space-time is described by the Lagrangian density

Lagrangian of ϕ^6 model

$$L = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - \frac{1}{2}\phi^2(\phi^2 - 1)^2$$

Where the potential which defines the self-interaction of the field, has the form is $V(\phi) = \frac{1}{2}\phi^2(\phi^2 - 1)^2$ and $\phi = \phi(t, x)$ is the field function. From static case of the Euler Lagrange equation with $\partial_t \phi = 0$, the equation can be reduced to first order differential equation as

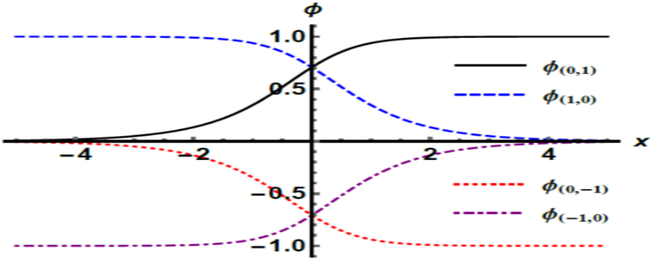
ϕ^6 model equation

$$\partial_t^2 \phi - \partial_x^2 \phi - \phi(\phi^2 - 1)(3\phi^2 - 1) = 0$$

, so we can find the solutions as

Kinks

$$\phi_{(0,1)}(x) = -\sqrt{\frac{1+tanh(x)}{2}},\ \phi_{(-1,0)}(x) = \sqrt{\frac{1-tanh(x)}{2}}$$



anti-kinks

$$\phi_{(1,0)}(x) = \sqrt{\frac{1+tanh(x)}{2}},\ \phi_{(0,-1)}(x) = -\sqrt{\frac{1-tanh(x)}{2}}$$

The sine-Gordon equation is a nonlinear hyperbolic partial differential equation involving the d'Alembert operator and the sine of the unknown function.

Sine Gordan Equation

$$\partial_{tt}\phi - \partial_{tt} + sin\phi = 0$$

Similarly, the static solution for the the equation will have the form

The solution of Sine Gordan Equation

$$\phi(x) = 4tan^{-1}(exp(\pm(x - x_0)))$$

where the field is centered around the point x_0.

Conclusion

We constructed simple solitons solutions in (1+1) space-time dimensional of scalar field theory with polynomial potential.

Thank You for Your Attention!