JINR

Bogoliubov Laboratory of Theoretical Physics

Numerical methods in theory of topological solitons

Uladzislau Klimashonak

Department of Theoretical Physics and Astrophysics,

Belarusian State University, Minsk 220004, Belarus

Supervisor: Yakov Shnir

BLTP, JINR, Dubna 141980, Moscow Region, Russia

Aims of the project:

- studying soliton solutions (kinks) in 1+1 dimension space-time in ϕ^4 and ϕ^6 models;
- studying the interaction between kink and massless fermions in ϕ^4 and ϕ^6;
- to obtain a numerical solution of the equations of motion of the kink coupled to the fermionic modes.

Lagrangian of the ϕ^4+ fermions model

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2}{2}(\phi^2 - 1)^2 + \overline{\Psi} i \hat{\partial} \Psi - g \overline{\Psi} \Psi \phi, (\mu = 0, 1)$$

where ϕ – scalar field;

g – coupling constant;

Ψ – massless fermion field, $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$;

$$\hat{\partial}\Psi = \gamma^\mu \partial_\mu \Psi,$$

$$\overline{\Psi} = \Psi^\dagger \gamma^0,$$

where γ^μ – Dirac matrices in 1+1 space-time:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Dimensionless variables:

$$[\phi] = 1, [\Psi] = m^{1/2}, [g] = m$$

$$\tilde{x} = mx; \quad \tilde{t} = mt; \quad \tilde{\Psi} = m^{-1/2}\Psi; \quad \tilde{g} = m^{-1} g; \quad \mathcal{L}(x, t) = m^2 \tilde{\mathcal{L}}(\tilde{x}, \tilde{t});$$

$$\tilde{\mathcal{L}} = \frac{1}{2}\left(\widetilde{\partial_\mu}\phi\right)^2 - \frac{1}{2}(\phi^2 - 1)^2 + \overline{\tilde{\Psi}} i \tilde{\hat{\partial}} \tilde{\Psi} - \tilde{g} \overline{\tilde{\Psi}} \tilde{\Psi} \phi$$

Equations Of Motion

$$S = \int d^2x \, \mathcal{L} = \int d^2x \, [\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}(\phi^2 - 1)^2 + \overline{\Psi} i \hat{\partial} \Psi - g \overline{\Psi} \Psi \phi]$$

The principle of least action: $\delta S = 0$.

Equations of motion:

$$\partial_\mu \partial_\mu \phi - 2\phi + 2\phi^3 + g\overline{\Psi}\Psi = 0;$$
$$i\gamma^\mu \partial_\mu \Psi - g\Psi\phi = 0.$$

The form of soliton-like stationary solutions with localized fermionic states: $\phi = \phi(x), \Psi = e^{-i\varepsilon t}\psi(x)$, where

$$\psi(x) = \begin{pmatrix} u_\varepsilon(x) \\ v_\varepsilon(x) \end{pmatrix}.$$

Normalization condition: $\int_{-\infty}^{\infty} dx \, [u_\varepsilon^2 + v_\varepsilon^2] = 1.$

After the substitution

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -2\phi + 2\phi^3 + 2g u_\varepsilon v_\varepsilon \\ \dfrac{du_\varepsilon}{dx} = \varepsilon v_\varepsilon - g\phi u_\varepsilon \\ \dfrac{dv_\varepsilon}{dx} = -\varepsilon u_\varepsilon + g\phi v_\varepsilon \end{cases}.$$

Numerical Problem

System of equations:

$$\begin{cases} \dfrac{du_\varepsilon}{dx} = \varepsilon v_\varepsilon - g\phi u_\varepsilon \\ \dfrac{dv_\varepsilon}{dx} = -\varepsilon u_\varepsilon + g\phi v_\varepsilon \\ \dfrac{d\phi'}{dx} = -2\phi + 2\phi^3 + 2g u_\varepsilon v_\varepsilon \\ \dfrac{d\phi}{dx} = \phi' \\ \int\limits_0^{+\infty} dx\,[u_\varepsilon^2 + v_\varepsilon^2] = 1/2 \end{cases}$$

Boundary conditions:

$$u_\varepsilon \Big|_{x\to+\infty} = v_\varepsilon \Big|_{x\to+\infty} = 0;$$

$$v_\varepsilon \Big|_{x=0} = 0;$$

$$u'_\varepsilon \Big|_{x=0} = 0;$$

$$\phi \Big|_{x=0} = 0;$$

$$\phi \Big|_{x\to+\infty} = 1.$$

The Shooting Method

System of ODE:

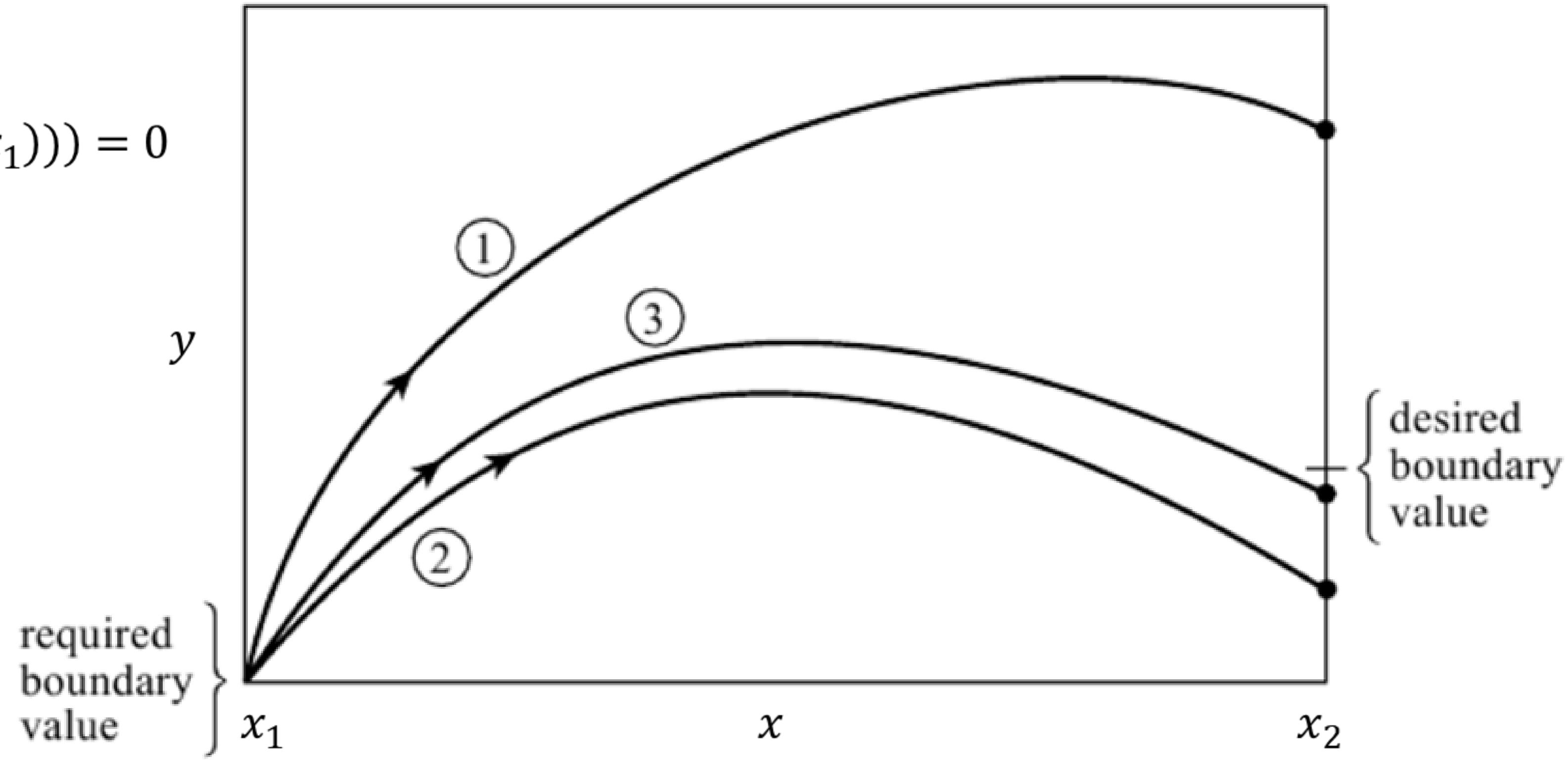
$$F_i(x, y_1, y_2, \dots, y_N) = 0, \qquad i = 1,2, \dots, N.$$

Boundary conditions:

$$B_{1j}\big(x_1, y_1(x_1), y_2(x_1), \dots, y_N(x_1)\big) = 0, \qquad j = 1, \dots, n_1;$$

$$B_{2j}\big(x_2, y_1(x_2), y_2(x_2), \dots, y_N(x_2)\big) = 0, \qquad j = 1, \dots, N - n_1.$$

$$B_{2j}(B_{1k}(x_1, y_1(x_1), y_2(x_1), \dots, y_N(x_1))) = 0$$



The Strategy Of Numerical Method

- 1-dimentional shooting method relatively $\phi'|_{x=0}$ for the subsystem

$$\begin{cases} \dfrac{d\phi'}{dx} = -2\phi + 2\phi^3 + 2gu_\varepsilon v_\varepsilon \\ \dfrac{d\phi}{dx} = \phi' \end{cases}$$

without coupling (g=0)

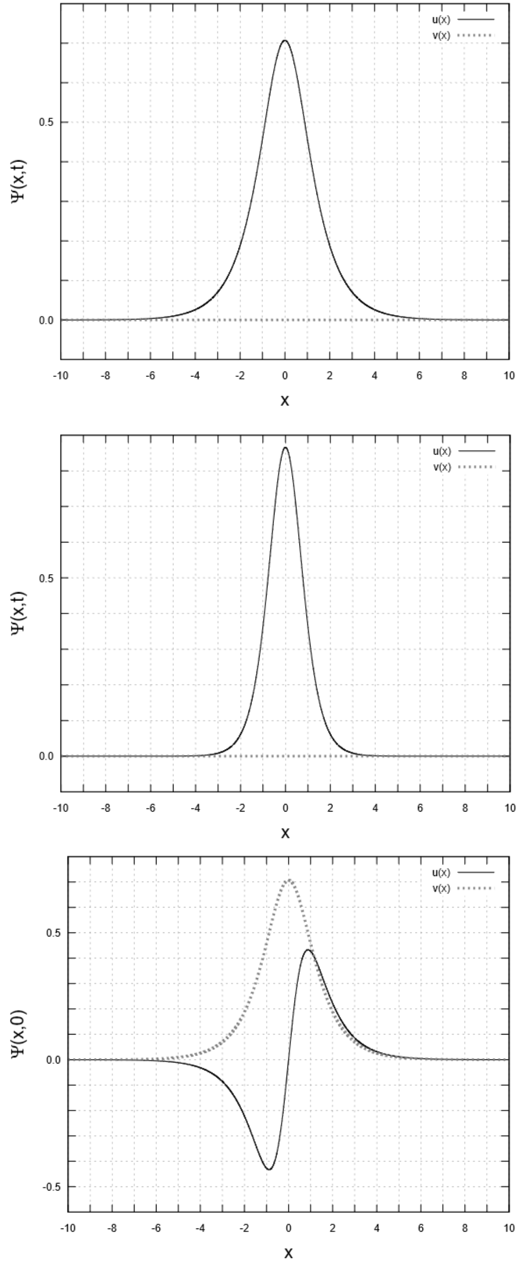
- 1-dimentional shooting method relatively $u_\varepsilon|_{x=0}$ for the subsystem

$$\begin{cases} \dfrac{du_\varepsilon}{dx} = \varepsilon v_\varepsilon - g\phi u_\varepsilon \\ \dfrac{dv_\varepsilon}{dx} = -\varepsilon u_\varepsilon + g\phi v_\varepsilon \end{cases}$$

- one step of Newton Method relatively ε for the equation $\int_0^{+\infty} dx\,[u_\varepsilon^2 + v_\varepsilon^2] = 1/2.$
- repeat iteration

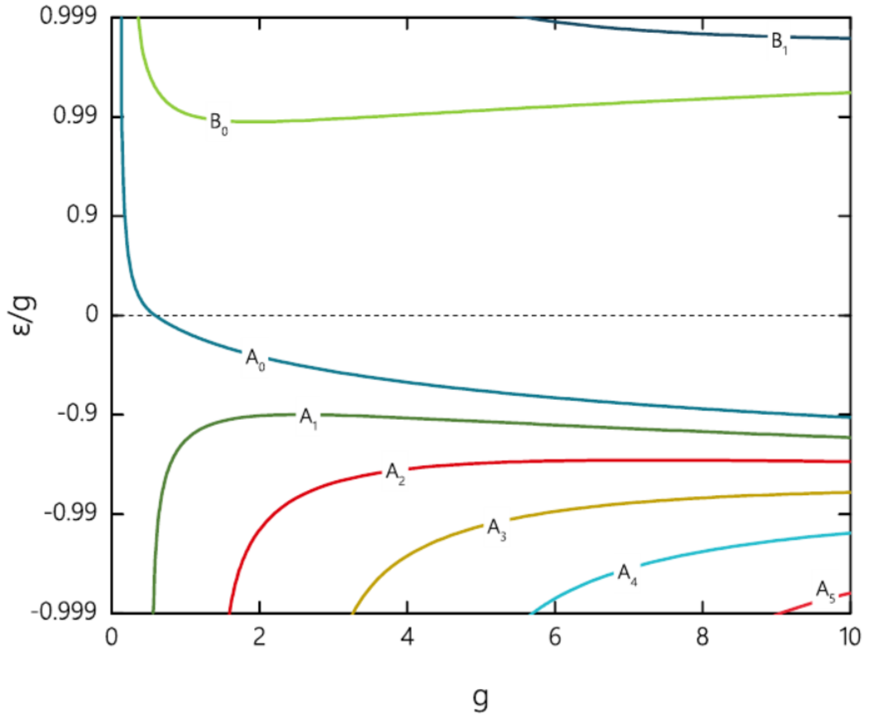
Analytical Solutions

1. The case $g = 1$:
   - $\varepsilon = 0$ – only one localized nondegenerate state:
$$\Psi^{g=1}_{\varepsilon=0}(x,t) = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\cosh x} \\ 0 \end{pmatrix};$$
   - $\varepsilon^2 \geq 1$ – the continuum of scattering states.
2. The case $g = 2$:
   - $\varepsilon = 0$ – localized nondegenerate state:
$$\Psi^{g=2}_{\varepsilon=0}(x,t) = \frac{\sqrt{3}}{2}\begin{pmatrix} \frac{1}{\cosh^2 x} \\ 0 \end{pmatrix};$$
   - $\varepsilon = \pm\sqrt{3}$ – localized nondegenerate state:
$$\Psi^{g=2}_{\varepsilon=\pm\sqrt{3}}(x,t) = e^{\mp i\sqrt{3}t}\begin{pmatrix} \pm\frac{\sqrt{3}}{2}\frac{\sinh x}{\cosh^2 x} \\ \frac{1}{2}\frac{1}{\cosh x} \end{pmatrix};$$
   - $\varepsilon^2 \geq 4$ – the continuum of scattering states.



Outlook

- to obtain the spectral flow
- to get the picture of distortion of the kink profile due to the influence of fermions
- to apply the finite-difference method for numerical solving of the above system



THANK YOU FOR ATTENTION!