# NUMERICAL METHOD IN THEORY OF TOPOLOGICAL SOLITONS


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# Outline

- Objectives of the project
- Introduction to the theory of solitons
- Models used to determine soliton solutions
- Boundary Conditions of models
- Finding about Topological Solitons
- Conclusion
- References
- Acknowledgements

# Objectives

- Investigate the Topological Soliton solutions using numerical methods models. To study the behaviour of the solitons under certain condition(boundary conditions) using integrable models of special potentials.
- constructing topological solitons by choosing the manifold of the field space in such a way that it gives rise to non-trivial mapping.

# Introduction

- The study of objects arises from non-trivial topological and their application to particle physics. This objects are called as topological solitons, they are solutions of non-linear PDEs, which are derived from a lagrangian system with non-trivial topology.
- The soliton waves are defines as a localised finite energy field configuration that travel without dispersion, which appear in non-linear theories in various space-time dimensions.[1]

# Solitons in nonlinear physics

Soliton: This is the solution of a nonlinear partial differential equation which represents a solitary travelling wave, which:

Is localized in space

Has a constant shape

Does not obey the superposition principle.

Examples:

- *Optical fibres, rogue waves in ocean - NLSE*
- *Josephson junctions - sine-Gordon model*
- *Lattice QCD - caloron solutions*
- *Superconductivity - Abrikosov-Nielsen-Olesen model.* [1]

# Topology

**Topological solitons**

- Kinks, domain walls, vortices in various models
- Skyrmions
- Hopons

**Non-Topological**

- Non-topological solitons
- KdV solutions
- Lump-solution in various polynomial models
- Oscillons

[1]

# Integrable model

- There are (3) integrable model used due to their special integrability and their non-trivial topology to determine the scattering . To begin with we can think of the Klein-Gordon field as simply a function on space time, also known as a scalar (non-interacting) field.
- To produce these type of integrable non-linear differential equation, one can modify the classical KG theory in a variety of ways to make it nonlinear (one can introduce interactions). The simplest way to do this is to add a potential term to the Lagrangian.

# Lagrangian equation and Principle of Least Action(s)

An introduction to a PDEs and integrable equation. Every mechanical systems is characterized by a definite function.

$$L(q, \dot{q}, t)$$

S= $\int_{t_1}^{t_2} L(q, \dot{q}, t)$dt ↔ this is known as Least Action, where $L(q, \dot{q}, t)$ is the lagrangian of the system. Taking the variance of this integration and solving using by part we obtain the equation of motion for $\delta S = 0$ and q = q + $\delta q$.

Then the equation of motion are given by $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$ [2]

# Types Of models

**Topological Solitons** :

The simplest stable lagrangian is

$$\boldsymbol{L} = (\partial_t \phi)^2 - (\partial_x \phi)^2 - V(\phi), \qquad 1.1$$

Which is derived from lagrangian density of the form

$\boldsymbol{L} = \partial_\mu \phi \, \partial^\mu \phi - V(\phi)$ ... $V(\phi)$ – minimum potential

- There exist different model according to the potential

$V(\phi) = \sin\,(\varphi)$ *or* $1-\cos(\varphi)$ Sine-Gordon Model 1.2

$V(\phi) = \frac{1}{2}(1-\varphi^2)^2$ $\phi^4$ –Model 1.3

$V(\phi) = \frac{1}{2}\varphi^2\,(1-\varphi^2)^2$ $\phi^6$ – Model [1] 1.4

# Types of Models(cont..)

- where ϕ(x, t)is a real scalar field. The potential V(ϕ), which defines the self-interaction of the field.

**in 1- dimensions:**

Completing the square of equation 1.1 leads to Bogomolnyi Equation

$$\left(\frac{\partial \phi}{\partial x}\right)^2 = V(\phi)$$

which is the static minimal- energy solution with the integration solution of this type:

$$\int \frac{d\phi}{\sqrt{2V(\phi)}} = \pm(x - x_0) \quad [4] \qquad 1.5$$

# Boundary Conditions

$$\boldsymbol{L} = (\partial_t\phi)^2 - (\partial_x\phi)^2 - V(\phi)$$

1. $\begin{cases} V(\phi) = 1 - \cos\phi \\ x \to \pm\infty \quad \leftrightarrow \phi \to \begin{cases} 0 \\ 2\pi \end{cases} \end{cases}$ these are the boundary conditions for equation 1.2

2. $\begin{cases} V(\phi) = \frac{1}{2}(1 - \phi^2)^2 \\ x \to \pm\infty \quad \leftrightarrow \phi \to \begin{cases} -1 \\ 1 \end{cases} \end{cases}$ these are the boundary condition for equation 1.3

3. $\begin{cases} V(\phi) = \frac{1}{2}\phi^2(1 - \phi^2)^2 \\ x \to \pm\infty \quad \leftrightarrow \phi \to \begin{cases} -1 \\ 0 \\ 1 \end{cases} \end{cases}$ these are the boundary condition for equation 1.4

- All this condition apply only if the conditions of $\phi$ gives the minimum potential for static solutions using the above Lagrange Density equation

[1]

# Sine-Gordon model [V(ϕ)=sin(ϕ)]

- SGE is a real-valued, hyperbolic, nonlinear wave equation defined on $\mathbb{R}^{1,1}$.
- Applying the boundary conditions of equation 1.2, then equations 1.1 becomes
  - General equation is $(\partial_t\varphi)^2 - (\partial_x\varphi)^2 - \sin(\varphi) = 0$,

$$\phi(x,t) = 2\tan^{-1}(\pm\Delta x - ct),$$

where t is in Lorentz terms of inertial frames,

But our interest is on static solution

$$(\partial_x\phi)^2 - \sin(\phi) = 0$$

$$\phi(x) = 2\tan^{-1}(\pm\Delta x) \begin{cases} + \text{ kink solution moving from } 0 \text{ to } 2\pi \\ -\text{unti} - \text{kink solution travelling from } 2\pi \text{ to } 0. \end{cases}$$

# Sine-Gorden model Kink and unti-kink solution and potential graph

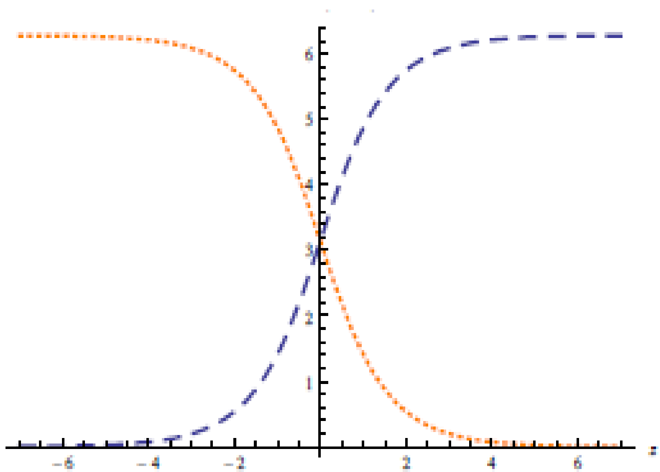

***Figure 1.1 a:***
The sine-Gordon Kink (red) and anti-kink (blue). [5]

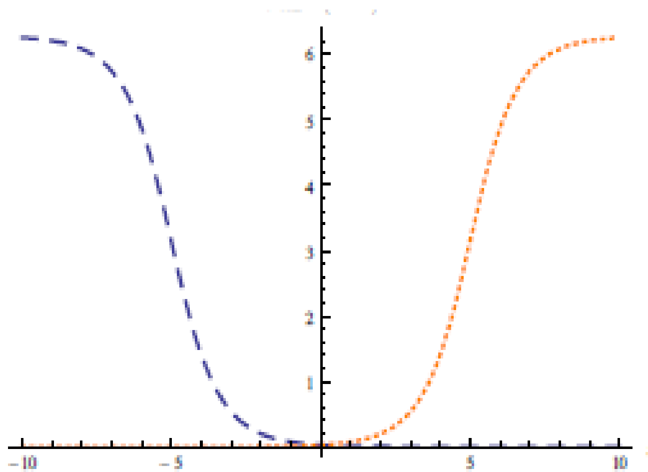

***Figure 1.1 b:***
The sine-Gordon Potential [5]

# ϕ$^4$ and ϕ$^6$ –Model

- Again applying the boundary conditions of equation 1.3 and 1.4,

  then equations 1.1 becomes :

  *interest is on static solution*

$$(\partial_x \varphi)^2 - \frac{1}{2}(1\text{-}\ \varphi^2)^2 = 0$$

$$\phi(x,t) = 2\tanh(\pm \Delta x),\ \begin{cases} +\ kink\ solution\ moving\ from\ -1\ to\ 1 \\ unti-kink\ solution\ travelling\ from\ 1\ to\ -1. \end{cases}$$

# $\phi^6$ – Model

Where $\phi^6$ – Model of potential: *solving the boundary condition we found that this scattering has minimum potential at the values of* $\phi \in (-1, 0, 1)$ *as* $x \to \pm\infty$.

$$V(\phi) = \frac{1}{2}\phi^2(1-\phi^2)^2$$

$$(\partial_x\phi)^2 - \frac{1}{2}\phi^2(1-\phi^2)^2 = 0$$

$$\phi(x) = \pm\sqrt{\frac{1}{2}(1 \mp \tanh \Delta x}$$

1-1 dimensional solution [4]

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# $\phi^4$ and $\phi^6$ –Model solutions and their potential

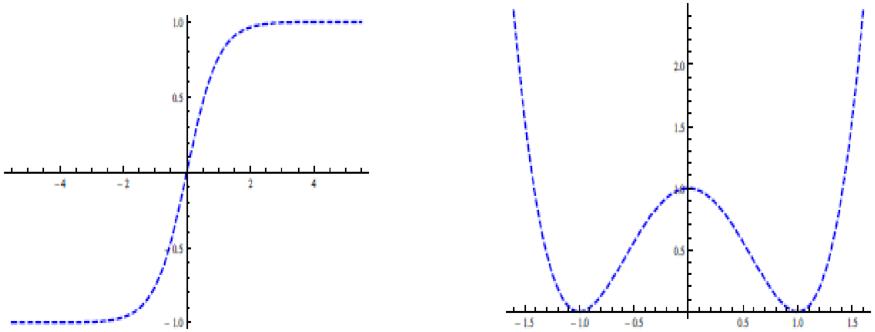

***Figure 1.2 a and b:***
kinks solutions and
Potential for the φ 4 model respectively [5]

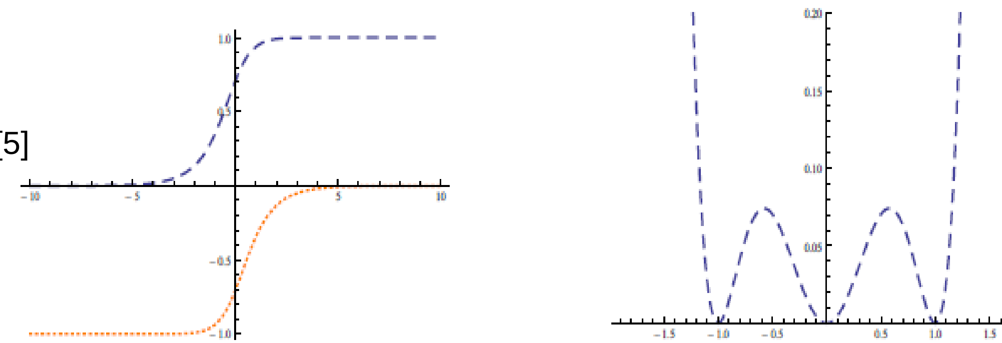

***Figure 1.3 a and b:***
kinks and unti-kinks and their
Potential for the φ 6 model [5]

# Example of kink and Unti-kink solutions travelling waves

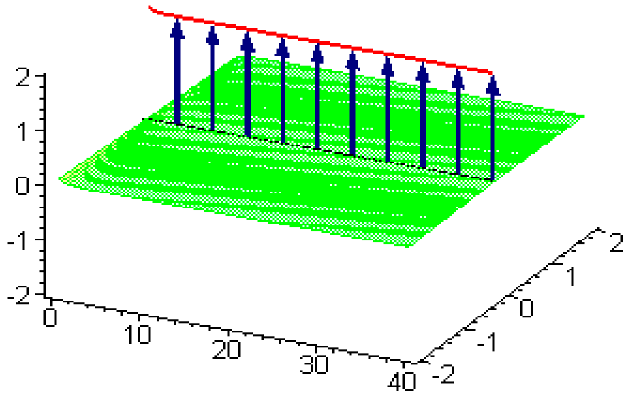


**Figure 1.4 a**

Traveling *kink* soliton represents propagating clockwise twist.[5]

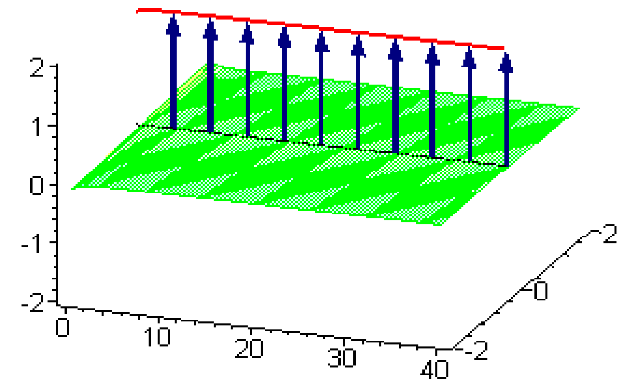


**Figure 1.4 b**

Traveling *anti-kink* soliton represents propagating counter-clockwise twist. [5]

# Conclusion

- We have shown that the solitons are solutions of Sine-Gordon and the $\phi^4$ and $\phi^6$kink model in (1+1) dimensions. or that solitons can be determined by PDEs of special Potential using linear equations transformed into nonlinear equations by adding interaction.
- These solitary spatiotemporal processes can serve as realistic models of nonlinear excitations in complex systems in physical sciences as well as in various living cellular structures, including intra–cellular ones (DNA, protein folding and microtubules) and inter–cellular ones (neural impulses and muscular contractions).

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## Thank You

## Questions ?

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